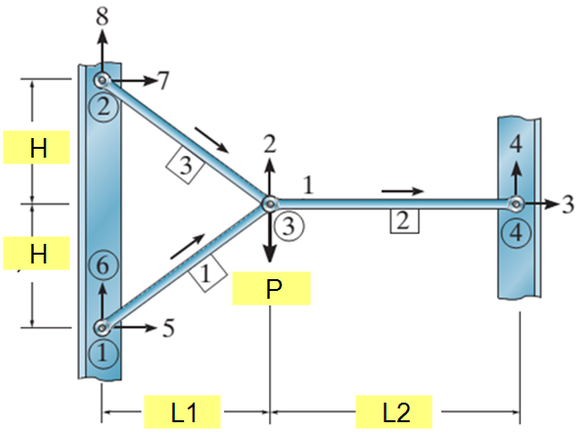
TE2_E08-01

Determinar o esforço axial em cada uma das barras indicadas na montagem ao lado.



Para todas as barras:

$E := 200$ GPa

$A := 1000$ mm²

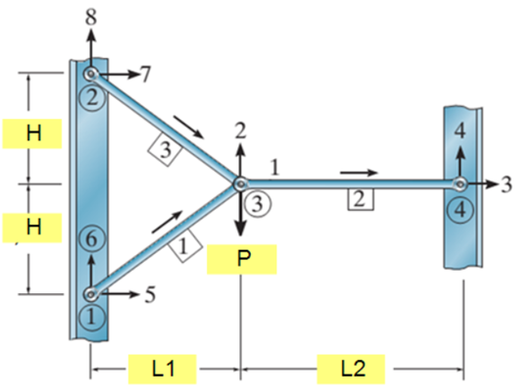
- Dados:

$P := 20$ kN

$L1 := 1,2$ m $\quad L2 := 1,8$ m $\quad H := 0,9$ m

Matrizes de Rigidez Elementares em Coordenadas Globais de Cada Barra:

Coordenadas (origem do sistema de eixos global no ponto "1"):



$x1 := 0$ $\quad y1 := 0$

$x2 := 0$ $\quad y2 := 2 \cdot H = 1,8$

$x3 := L1 = 1,2$ $\quad y3 := H = 0,9$

$x4 := L1 + L2 = 3$ $\quad y4 := H = 0,9$

$$[k] = \frac{AE}{L}\begin{bmatrix} \lambda_x^2 & \lambda_x\lambda_y & -\lambda_x^2 & -\lambda_x\lambda_y \\ \lambda_x\lambda_y & \lambda_y^2 & -\lambda_x\lambda_y & -\lambda_y^2 \\ -\lambda_x^2 & -\lambda_x\lambda_y & \lambda_x^2 & \lambda_x\lambda_y \\ -\lambda_x\lambda_y & -\lambda_y^2 & \lambda_x\lambda_y & \lambda_y^2 \end{bmatrix} \begin{matrix} N_x \\ N_y \\ F_x \\ F_y \end{matrix}$$

$$\begin{matrix} N_x & N_y & F_x & F_y \end{matrix}$$

Matriz de Rigidez Elementar em Coord. Globais da Barra 1: $\quad EA := 1$

$L1 := \sqrt{(x3 - x1)^2 + (y3 - y1)^2} = 1,5$ $\quad \lambda x1 := \frac{x3 - x1}{L1} = 0,8$ $\quad \lambda y1 := \frac{y3 - y1}{L1} = 0,6$

$$\begin{matrix} & & & & [\,5 & 6 & 1 & 2\,] \end{matrix}$$

$$k1 := \frac{EA}{L1} \cdot \begin{bmatrix} \lambda x1^2 & \lambda x1 \cdot \lambda y1 & -\lambda x1^2 & -\lambda x1 \cdot \lambda y1 \\ \lambda x1 \cdot \lambda y1 & \lambda y1^2 & -\lambda x1 \cdot \lambda y1 & -\lambda y1^2 \\ -\lambda x1^2 & -\lambda x1 \cdot \lambda y1 & \lambda x1^2 & \lambda x1 \cdot \lambda y1 \\ -\lambda x1 \cdot \lambda y1 & -\lambda y1^2 & \lambda x1 \cdot \lambda y1 & \lambda y1^2 \end{bmatrix} = \begin{bmatrix} \frac{32}{75} & \frac{8}{25} & -\frac{32}{75} & -\frac{8}{25} \\ \frac{8}{25} & \frac{6}{25} & -\frac{8}{25} & -\frac{6}{25} \\ -\frac{32}{75} & -\frac{8}{25} & \frac{32}{75} & \frac{8}{25} \\ -\frac{8}{25} & -\frac{6}{25} & \frac{8}{25} & \frac{6}{25} \end{bmatrix} \begin{matrix} 5 \\ 6 \\ 1 \\ 2 \end{matrix}$$

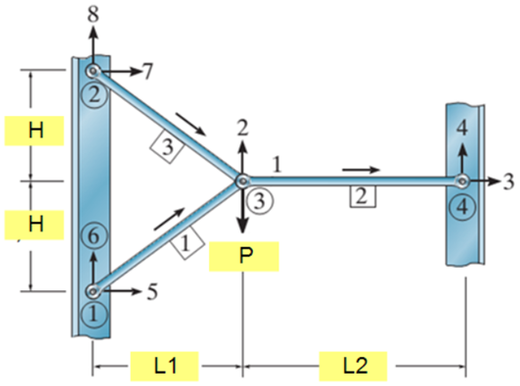
Matriz de Rigidez Elementar em Coord. Globais da Barra 2:

$L2 := \sqrt{(x4 - x3)^2 + (y4 - y3)^2} = 1,8$ $\quad \lambda x2 := \frac{x4 - x3}{L2} = 1$ $\quad \lambda y2 := \frac{y4 - y3}{L2} = 0$

$$\begin{matrix} & & & & [\,1 & 2 & 3 & 4\,] \end{matrix}$$

$$k2 := \frac{EA}{L2} \cdot \begin{bmatrix} \lambda x2^2 & \lambda x2 \cdot \lambda y2 & -\lambda x2^2 & -\lambda x2 \cdot \lambda y2 \\ \lambda x2 \cdot \lambda y2 & \lambda y2^2 & -\lambda x2 \cdot \lambda y2 & -\lambda y2^2 \\ -\lambda x2^2 & -\lambda x2 \cdot \lambda y2 & \lambda x2^2 & \lambda x2 \cdot \lambda y2 \\ -\lambda x2 \cdot \lambda y2 & -\lambda y2^2 & \lambda x2 \cdot \lambda y2 & \lambda y2^2 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & 0 & -\frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{5}{9} & 0 & \frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Matriz de Rigidez Elementar em Coord. Globais da Barra 3:



$$L3 := \sqrt{(x3 - x2)^2 + (y3 - y2)^2} = 1,5$$

$$\lambda x3 := \frac{x3 - x2}{L3} = 0,8 \qquad \lambda y3 := \frac{y3 - y2}{L3} = -0,6$$

$$[7\ 8\ 1\ 2]$$

$$k3 := \frac{EA}{L3} \cdot \begin{bmatrix} \lambda x3^2 & \lambda x3 \cdot \lambda y3 & -\lambda x3^2 & -\lambda x3 \cdot \lambda y3 \\ \lambda x3 \cdot \lambda y3 & \lambda y3^2 & -\lambda x3 \cdot \lambda y3 & -\lambda y3^2 \\ -\lambda x3^2 & -\lambda x3 \cdot \lambda y3 & \lambda x3^2 & \lambda x3 \cdot \lambda y3 \\ -\lambda x3 \cdot \lambda y3 & -\lambda y3^2 & \lambda x3 \cdot \lambda y3 & \lambda y3^2 \end{bmatrix} = \begin{bmatrix} \frac{32}{75} & -\frac{8}{25} & -\frac{32}{75} & \frac{8}{25} \\ -\frac{8}{25} & \frac{6}{25} & \frac{8}{25} & -\frac{6}{25} \\ -\frac{32}{75} & \frac{8}{25} & \frac{32}{75} & -\frac{8}{25} \\ \frac{8}{25} & -\frac{6}{25} & -\frac{8}{25} & \frac{6}{25} \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix}$$

Montagem da Matriz de Rigidez da Estrutura como um Todo:

$$KG := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Espalhamento de "k1":

$$[5\ 6\ 1\ 2]$$

$$k1 = \begin{bmatrix} \frac{32}{75} & \frac{8}{25} & -\frac{32}{75} & -\frac{8}{25} \\ \frac{8}{25} & \frac{6}{25} & -\frac{8}{25} & -\frac{6}{25} \\ -\frac{32}{75} & -\frac{8}{25} & \frac{32}{75} & \frac{8}{25} \\ -\frac{8}{25} & -\frac{6}{25} & \frac{8}{25} & \frac{6}{25} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \\ 2 \end{bmatrix} \quad ===> \quad KG := \begin{bmatrix} k1_{3\,3} & k1_{3\,4} & 0\ 0 & k1_{3\,1} & k1_{3\,2} & 0\ 0 \\ k1_{4\,3} & k1_{4\,4} & 0\ 0 & k1_{4\,1} & k1_{4\,2} & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \\ k1_{1\,3} & k1_{1\,4} & 0\ 0 & k1_{1\,1} & k1_{1\,2} & 0\ 0 \\ k1_{2\,3} & k1_{2\,4} & 0\ 0 & k1_{2\,1} & k1_{2\,2} & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \end{bmatrix} = \begin{bmatrix} \frac{32}{75} & \frac{8}{25} & 0\ 0 & -\frac{32}{75} & -\frac{8}{25} & 0\ 0 \\ \frac{8}{25} & \frac{6}{25} & 0\ 0 & -\frac{8}{25} & -\frac{6}{25} & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \\ -\frac{32}{75} & -\frac{8}{25} & 0\ 0 & \frac{32}{75} & \frac{8}{25} & 0\ 0 \\ -\frac{8}{25} & -\frac{6}{25} & 0\ 0 & \frac{8}{25} & \frac{6}{25} & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0\ 0 & 0 & 0 & 0\ 0 \end{bmatrix}$$

Espalhamento de "k2":

$$[1\ 2\ 3\ 4] \qquad [1\ 2\ 3\ 4]$$

$$k2 = \begin{bmatrix} \frac{5}{9} & 0 & -\frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{5}{9} & 0 & \frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad ===> \quad KG := \begin{bmatrix} k1_{3\,3} + k2_{1\,1} & k1_{3\,4} + k2_{1\,2} & k2_{1\,3} & k2_{1\,4} & k1_{3\,1} & k1_{3\,2} & 0\ 0 \\ k1_{4\,3} + k2_{2\,1} & k1_{4\,4} + k2_{2\,2} & k2_{2\,3} & k2_{2\,4} & k1_{4\,1} & k1_{4\,2} & 0\ 0 \\ k2_{3\,1} & k2_{3\,2} & k2_{3\,3} & k2_{3\,4} & 0 & 0 & 0\ 0 \\ k2_{4\,1} & k2_{4\,2} & k2_{4\,3} & k2_{4\,4} & 0 & 0 & 0\ 0 \\ k1_{1\,3} & k1_{1\,4} & 0 & 0 & k1_{1\,1} & k1_{1\,2} & 0\ 0 \\ k1_{2\,3} & k1_{2\,4} & 0 & 0 & k1_{2\,1} & k1_{2\,2} & 0\ 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \end{bmatrix}$$

$$KG = \begin{bmatrix} \frac{221}{225} & \frac{8}{25} & -\frac{5}{9} & 0 & -\frac{32}{75} & -\frac{8}{25} & 0 & 0 \\ \frac{8}{25} & \frac{6}{25} & 0 & 0 & -\frac{8}{25} & -\frac{6}{25} & 0 & 0 \\ -\frac{5}{9} & 0 & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{32}{75} & -\frac{8}{25} & 0 & 0 & \frac{32}{75} & \frac{8}{25} & 0 & 0 \\ -\frac{8}{25} & -\frac{6}{25} & 0 & 0 & \frac{8}{25} & \frac{6}{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Espalhamento de "k3":

$$\begin{bmatrix} 7 & 8 & 1 & 2 \end{bmatrix}$$

$$k3 = \begin{bmatrix} \frac{32}{75} & -\frac{8}{25} & -\frac{32}{75} & \frac{8}{25} \\ -\frac{8}{25} & \frac{6}{25} & \frac{8}{25} & -\frac{6}{25} \\ -\frac{32}{75} & \frac{8}{25} & \frac{32}{75} & -\frac{8}{25} \\ \frac{8}{25} & -\frac{6}{25} & -\frac{8}{25} & \frac{6}{25} \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{===>}\ KG := \begin{bmatrix} k1_{3\,3} + k2_{1\,1} + k3_{3\,3} & k1_{3\,4} + k2_{1\,2} + k3_{3\,4} & k2_{1\,3} & k2_{1\,4} & k1_{3\,1} & k1_{3\,2} & k3_{3\,1} & k3_{3\,2} \\ k1_{4\,3} + k2_{2\,1} + k3_{4\,3} & k1_{4\,4} + k2_{2\,2} + k3_{4\,4} & k2_{2\,3} & k2_{2\,4} & k1_{4\,1} & k1_{4\,2} & k3_{4\,1} & k3_{4\,2} \\ k2_{3\,1} & k2_{3\,2} & k2_{3\,3} & k2_{3\,4} & 0 & 0 & 0 & 0 \\ k2_{4\,1} & k2_{4\,2} & k2_{4\,3} & k2_{4\,4} & 0 & 0 & 0 & 0 \\ k1_{1\,3} & k1_{1\,4} & 0 & 0 & k1_{1\,1} & k1_{1\,2} & 0 & 0 \\ k1_{2\,3} & k1_{2\,4} & 0 & 0 & k1_{2\,1} & k1_{2\,2} & 0 & 0 \\ k3_{1\,3} & k3_{1\,4} & 0 & 0 & 0 & 0 & k3_{1\,1} & k3_{1\,2} \\ k3_{2\,3} & k3_{2\,4} & 0 & 0 & 0 & 0 & k3_{2\,1} & k3_{2\,2} \end{bmatrix}$$

$$KG = \begin{bmatrix} \frac{317}{225} & 0 & -\frac{5}{9} & 0 & -\frac{32}{75} & -\frac{8}{25} & -\frac{32}{75} & \frac{8}{25} \\ 0 & \frac{12}{25} & 0 & 0 & -\frac{8}{25} & -\frac{6}{25} & \frac{8}{25} & -\frac{6}{25} \\ -\frac{5}{9} & 0 & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{32}{75} & -\frac{8}{25} & 0 & 0 & \frac{32}{75} & \frac{8}{25} & 0 & 0 \\ -\frac{8}{25} & -\frac{6}{25} & 0 & 0 & \frac{8}{25} & \frac{6}{25} & 0 & 0 \\ -\frac{32}{75} & \frac{8}{25} & 0 & 0 & 0 & 0 & \frac{32}{75} & -\frac{8}{25} \\ \frac{8}{25} & -\frac{6}{25} & 0 & 0 & 0 & 0 & -\frac{8}{25} & \frac{6}{25} \end{bmatrix}$$

$$K11 := \begin{bmatrix} KG_{1\,1} & KG_{1\,2} \\ KG_{2\,1} & KG_{2\,2} \end{bmatrix} = \begin{bmatrix} \frac{317}{225} & 0 \\ 0 & \frac{12}{25} \end{bmatrix}$$

$$K21 := \begin{bmatrix} KG_{3\,1} & KG_{3\,2} \\ KG_{4\,1} & KG_{4\,2} \\ KG_{5\,1} & KG_{5\,2} \\ KG_{6\,1} & KG_{6\,2} \\ KG_{7\,1} & KG_{7\,2} \\ KG_{8\,1} & KG_{8\,2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{9} & 0 \\ 0 & 0 \\ -\frac{32}{75} & -\frac{8}{25} \\ -\frac{8}{25} & -\frac{6}{25} \\ -\frac{32}{75} & \frac{8}{25} \\ \frac{8}{25} & -\frac{6}{25} \end{bmatrix}$$

$$Qk := \begin{bmatrix} 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ -20 \end{bmatrix}$$

$$K12 := \begin{bmatrix} KG_{1\,3} & KG_{1\,4} & KG_{1\,5} & KG_{1\,6} & KG_{1\,7} & KG_{1\,8} \\ KG_{2\,3} & KG_{2\,4} & KG_{2\,5} & KG_{2\,6} & KG_{2\,7} & KG_{2\,8} \end{bmatrix} = \begin{bmatrix} -\frac{5}{9} & 0 & -\frac{32}{75} & -\frac{8}{25} & -\frac{32}{75} & \frac{8}{25} \\ 0 & 0 & -\frac{8}{25} & -\frac{6}{25} & \frac{8}{25} & -\frac{6}{25} \end{bmatrix}$$

$$Qu := \begin{bmatrix} Q3 \\ Q4 \\ Q5 \\ Q6 \\ Q7 \\ Q8 \end{bmatrix} \qquad Dk := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$K22 := \begin{bmatrix} KG_{3\,3} & KG_{3\,4} & KG_{3\,5} & KG_{3\,6} & KG_{3\,7} & KG_{3\,8} \\ KG_{4\,3} & KG_{4\,4} & KG_{4\,5} & KG_{4\,6} & KG_{4\,7} & KG_{4\,8} \\ KG_{5\,3} & KG_{5\,4} & KG_{5\,5} & KG_{5\,6} & KG_{5\,7} & KG_{5\,8} \\ KG_{6\,3} & KG_{6\,4} & KG_{6\,5} & KG_{6\,6} & KG_{6\,7} & KG_{6\,8} \\ KG_{7\,3} & KG_{7\,4} & KG_{7\,5} & KG_{7\,6} & KG_{7\,7} & KG_{7\,8} \\ KG_{8\,3} & KG_{8\,4} & KG_{8\,5} & KG_{8\,6} & KG_{8\,7} & KG_{8\,8} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{32}{75} & \frac{8}{25} & 0 & 0 \\ 0 & 0 & \frac{8}{25} & \frac{6}{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{32}{75} & -\frac{8}{25} \\ 0 & 0 & 0 & 0 & -\frac{8}{25} & \frac{6}{25} \end{bmatrix}$$

Solução do Sistema de Equações:

Solução do Sistema de Equações: Qk = K11*Du + K12*Dk (1ª Linha)
Qu = K21*Du + K22*Dk (2ª Linha)

Resolvendo a 1ª Linha: Du = K11^-1 * (Qk - K12*Dk) como Dk = 0 ---> Du = K11^-1 * Qk

$$Du := K11^{-1} \cdot Qk = \begin{bmatrix} 0 \\ -41,6667 \end{bmatrix}$$

$$Du_com_AE := K11^{-1} \cdot Qk \cdot \frac{1}{A \cdot E} = \begin{bmatrix} 0,0000 \\ -0,00020833 \end{bmatrix}$$

$D1 := Du_1 = 0$ $D2 := Du_2 = -41,6667$ $D3 := 0$ $D4 := 0$ $D5 := 0$ $D6 := 0$ $D7 := 0$ $D8 := 0$

Resolvendo a 2ª Linha: Qu = K21*Du + K22*Dk como Dk = 0 ---> Du = K21*Du

$$Qu := K21 \cdot Du = \begin{bmatrix} 0 \\ 0 \\ 13,3333 \\ 10 \\ -13,3333 \\ 10 \end{bmatrix}$$

* Reações de apoio conforme a convenção de Green nos graus de liberdade 3,4,5,6,7 e 8 respectivamente

Esforços Axiais nas Barras:

$$q_F = \frac{AE}{L}[-\lambda_x \quad -\lambda_y \quad \lambda_x \quad \lambda_y] \begin{Bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Fx} \\ D_{Fy} \end{Bmatrix}$$

Na Barra 1: [5 6 1 2]

$$N1 := \frac{EA}{L1} \cdot \left(\left([-\lambda x1 \quad -\lambda y1 \quad \lambda x1 \quad \lambda y1] \cdot \begin{bmatrix} D5 \\ D6 \\ D1 \\ D2 \end{bmatrix} \right)_1 \right) = -16,667$$

Na Barra 2: [1 2 3 4]

$$N2 := \frac{EA}{L2} \cdot \left(\left([-\lambda x2 \quad -\lambda y2 \quad \lambda x2 \quad \lambda y2] \cdot \begin{bmatrix} D1 \\ D2 \\ D3 \\ D4 \end{bmatrix} \right)_1 \right) = 0$$

Na Barra 3: [7 8 1 2]

$$N3 := \frac{EA}{L3} \cdot \left(\left([-\lambda x3 \quad -\lambda y3 \quad \lambda x3 \quad \lambda y3] \cdot \begin{bmatrix} D7 \\ D8 \\ D1 \\ D2 \end{bmatrix} \right)_1 \right) = 16,6667$$